Dimension of branching processes and self-organized criticality

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Branching processes and their application as a model of self-organized criticality are briefly reviewed. The critical dimension for this model is calculated. The differences between our result and similar ones on polymers and percolation are explained. We discuss semiquantitatively why the critical dimension of a model of self-organized criticality that includes the oscillation of the sandpile around its critical value would be different, perhaps even infinite. Finally, we conjecture that our mathematical results are more general than they seem.

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I. INTRODUCTION

Branching processes are a special kind of Markov process where the set of states is the set of natural numbers. If we denote by $P_{k\to j}(t)$ the probability of going from the state k to the state j in a time t, then a Markov process is a branching process if

$$P_{k\to j}(t)=\sum_{\substack{l_1,\ldots,l_k\\ l_1+\cdots+l_k=j}} P_{1\to l_1}(t)\cdots P_{1\to l_k}(t)\ . \tag{1}$$

It is convenient to think of $P_{1\to l_1}(t)$ as the probability that a living being (who reproduces asexually) will have l_1 descendents at time t. Then (1) says that these probabilities do not depend on the presence or absence of other individuals. In fact, branching processes were introduced to study statistical matters concerning family trees in the 19th century [1,2] and have since had other applications in genetics and in other branches of biology [3–5].

An important application of branching processes in physics has been the study of chain reactions in a bomb or in a nuclear reactor [6–8]. In this case the neutrons take the place of the living beings.

More recently [9,10], branching processes have been applied to the study of self-organized criticality [11,12]. In these models grains of sand take the place of neutrons. Time can be taken to be discrete [9] or continuous [10], but as $t\to\infty$ both choices become equivalent [10]. Since in this paper we will be dealing with asymptotic properties, we will use the discrete or the continuous time model according to our convenience.

In the continuous time model for self-organized criticality, the probability for the grain of sand to get stuck during an infinitesimal amount of time dt is $\mu\,dt$. The probability for the grain of sand to kick another grain of sand into motion is $\mu\,dt$. The probability for none of the above to happen is $1-2\mu\,dt$. We will call these events "death," "reproduction," and "nothing," respectively.

The last paragraph remains true if "continuous" is substituted by "discrete," "an infinitesimal amount of time dt" by "a unit of discrete time Δt," and, everywhere else, "dt" by "Δt."

Note that in both models the expected number of descendents m is 1, since the probabilities of death and reproduction cancel each other out. This condition $m=1$ is known as "criticality," and it is one of the ingredients of the self-organized criticality conjecture.

In the continuous time model one can write a differential equation for the transition probabilities, known as the Chapman-Kolmogorov equation and solve it [10,13]. The solutions are

$$\begin{aligned}P_{1\to 0}(t)&=\frac{\mu t}{1+\mu t}\ ,\\ P_{1\to j}(t)&=\frac{1}{\mu^2t^2}\left[\frac{\mu t}{1+\mu t}\right]^{j+1},\quad j\neq 0\ .\end{aligned}\tag{2}$$

II. CRITICAL DIMENSION

After this background we can introduce the question addressed in this article. If Eq. (1) is regarded not as a definition but as a condition verified by an actual process, then in general it will be an approximation equivalent to neglecting the interactions between the different individuals. If the individuals are asexual beings, the interactions will be competition for food. If they are sexual beings, there will also be competition to mate. If the individuals are grains of sand, then we have argued [10] that there will be interaction if their paths cross in space (crossing in space is sufficient for interaction; there does not have to be crossing in space-time). Here we ask the reader to picture the paths left by all the grains of sand that took part in an avalanche and notice that they form a tree whose root is at the tip of the sandpile and which grows downward.

This happens more often the lower the dimension. To

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have an image of this, think of the branches of a tree, which do not cross each other, but their shadows, which are in a two-dimensional space, do. Conversely, if the dimension of the embedding space is high enough, the branches will meet with probability zero.

To be more precise, let D_0 be the dimension of the tree and let d be the dimension of the embedding space. Then the intersection of two parts of the tree, which have dimension $2D_0$ themselves, will generically have a dimension $2D_0-d$. If $d>2D_0$, the intersection's negative dimension should be interpreted as the exponent by which the number of sites of the intersection scales if we are in a discrete space and we are letting the spacing of the lattice go to zero. Thus, for $d>2D_0$ and for infinitely large trees, the proportion of their volume that intersects each other is zero and, for that reason, the noninteracting model for which Eq. (1) holds becomes exact. The dimension $2D_0$ is called the "critical dimension."

The computation of the probability density that the intersection of the trees will affect a certain fraction of the volume can in principle be carried out for trees of finite sizes. But for infinitely large trees the said probability density becomes a Dirac δ function and the concept of critical dimension can be well defined. These remarks will become clearer in the course of the calculation to follow.

First, let us define the probabilities $\{P_j'(t)\}_{j=1}^{\infty}$, as the probabilities $\{P_j(t)\}_{j=1}^{\infty}$ provided that the tree exists, i.e., they satisfy

$$\sum_{j=1}^{\infty} P_j'(t)=1 \tag{3}$$

and

$$\frac{P_j'(t)}{P_j(t)}=\text{const} . \tag{4}$$

Explicit calculation yields

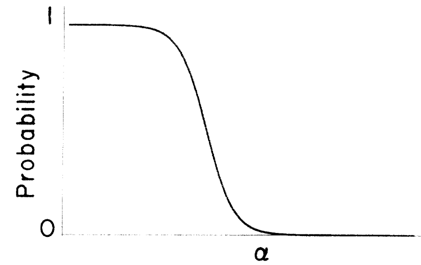

FIG. 1. Plot of the proportion of trees that grow at least as fast as t^{α}.

$$P_j'(t)=\frac{P_j(t)}{[1-P_0(t)]}=\frac{1}{\mu t}\left[\frac{\mu t}{1+\mu t}\right]^j . \tag{5}$$

Consider the sum $\sum_{j=kt^{\alpha}}^{\infty} P_j'(t)$, where k is some positive real number. This sum is the probability that at time t a tree has grown as fast or faster than kt^{α} divided by the probability that that tree exists. Whether the lower limit in the sum is read as "j is equal to the closest integer to kt^{α}" or "j is equal to the highest integer smaller than kt^{α}" or some other definition along these lines is going to be irrelevant as $t\to\infty$. It is clear that for $\alpha=0$ the value of the sum is 1, for $\alpha=\infty$ the value of the sum is 0, and that it decreases monotonically in between, as sketched in Fig. 1. Now if in the limit $t\to\infty$ the number of grains of sand in the avalanche is going to scale as $j\sim t^{\alpha_0}$, then the limit $\lim_{t\to\infty}\sum_{j=kt^{\alpha}}^{\infty}P_j'(t)$ will jump from 0 to 1 at α_0. In other words, the function sketched in Fig. 1 develops a singularity as $t\to\infty$, as shown in Fig. 2. We remind the reader that the Hausdorff-Besicovitch dimension can also be defined as the value at which a certain function develops a singularity when a limit of analogous significance to our $t\to\infty$ limit is taken [14].

We now show the calculation:

$$\begin{aligned}
\lim_{t\to\infty}\sum_{j=kt^{\alpha}}^{\infty} P_j'(t) &= \lim_{t\to\infty}\sum_{j=kt^{\alpha}}^{\infty}\frac{1}{\mu t}\left[\frac{\mu t}{1+\mu t}\right]^j \\
&= \lim_{t\to\infty}\frac{1}{\mu t}\left[\frac{\mu t}{1+\mu t}\right]^{kt^{\alpha}}\sum_{j=0}^{\infty}\left[\frac{\mu t}{1+\mu t}\right]^j = \lim_{t\to\infty}\frac{1}{\mu t}\left[\frac{\mu t}{1+\mu t}\right]^{kt^{\alpha}}\frac{1}{1-\dfrac{\mu t}{1+\mu t}} \\
&= \lim_{t\to\infty}\frac{1+\mu t}{\mu t}\left[\frac{\mu t}{1+\mu t}\right]^{kt^{\alpha}} = \lim_{t\to\infty}\left[\frac{\mu t}{1+\mu t}\right]^{kt^{\alpha}} = \lim_{t\to\infty}\exp\left[kt^{\alpha}\ln\left[1-\frac{1}{1+\mu t}\right]\right] \\
&= \lim_{t\to\infty}\exp\left[\frac{-kt^{\alpha}}{\mu t}\right] \\
&= \begin{cases} 0, & \alpha>1 \\ e^{-(k/\mu)}, & \alpha=1 \\ 1, & \alpha<1 . \end{cases}
\end{aligned} \tag{6}$$

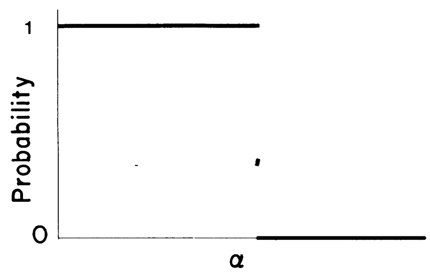

FIG. 2. As $t \to \infty$, the preceding plot develops a jump from 1 to 0. It shows that all trees grow like t raised to the number at which the discontinuity takes place.

This shows that the number of branches grows proportionally to time. Thus it follows that, for infinitely old avalanches, the number of grains is proportional to time. How does this relate to the spatial dimension of the corresponding trees? If the grains of sand move with some constant velocity along some preferred direction, e.g., the direction of the gradient of the height of the sandpile then "proportional to time" can be substituted by "proportional to the length along the longitudinal direction." Since, for the tree image, "number of grains" "means number of branches," we have the result that the number of branches of these infinitely old trees is proportional to their height. In other words, if they are embedded in a two-dimensional cone, the average volume of tree per unit area will be constant, or the tree is two dimensional and the critical dimension is 4.

What we have just described is a "directed model," i.e., there is some preferred direction. What if the model is not only homogeneous but also isotropic? The only propagation possible with both these constraints is diffusive, so that the distance to the origin is going to be now proportional to $\sqrt{t}$. Since the volume of the tree grows as t^2, the tree is now four dimensional and the critical dimension is 8.

These results agree with numerical research done both on the problem of percolation and on a computer model of avalanches. Obukhov [15] has found the critical dimension of a model of self-organized criticality to be 4, but numerical studies on a cellular automaton model of Christensen, Fogedby, and Jensen [16] show that its critical dimension is at least 6.

III. RELATION TO POLYMERS

The reader familiar with percolation or polymers might think that the preceding calculations are a derivation of a particular case of the result obtained by Zimm and Stockmayer in 1949 [17]. This is not so. The result obtained by these authors was that the dimension of a polymer obtained by randomly connecting chains is 4. More precisely, they considered all the polymers that can be built with a fixed number of segments n, which are joined in such a way that at most z of them, where z is a positive integer, can meet at a point. If the orientation that the segments can take is random, then the average radius of gyration of the polymer R, scales as $n^{1/4}$, i.e., $\lim_{n\to\infty} R(n) \propto n^{1/4}$. *In this average all polymers are equally likely.* This result applies directly to percolation in a Bethe lattice (or Cayley tree). There, all "animals" made of a fixed number n of bonds have the same perimeter, and thus [18] the same probability. Therefore, these lattice animals have dimension 4.

We have also been computing the dimension of a typical tree. But, as we shall see, what we mean by typical is not the same as what was meant by Zimm and Stockmayer. An example from the discrete time model will suffice. Consider the two trees of Fig. 3. They are both made of three elements. To calculate their probabilities to occur, we have to multiply the probabilities of the events taking place at each generation change. For the first tree the events are "nothing" in the first change, "nothing" in the second change, and "death" in the third change. The product of the respective probabilities is $(1-2\mu)^2\mu$. For the second tree we have reproduction in the first change and two "deaths" in the second change. The product of the respective probabilities is μ^3. Thus, in contrast to the polymers of fixed n, our trees of fixed n carry different weights and there is no reason why averages calculated with different weights should be the same. In fact, we shall see in Sec. IV that for noncritical branching processes the dimension is not 4.

It follows from the comparison between the results on polymers and ours that "dimension" or "mean square radius" are measurable quantities not of a chemical species but of a set of them in which different versions appear with different probabilities. Zimm and Stockmayer's choice of probabilities is mathematically simple and corresponds to homogeneous conditions of synthesis. In certain environments, however, such as biological ones, the synthesis might be taking place in the presence of gradients of concentrations, and this could imply a dimension or a mean square radius different from the ones predicted by Zimm and Stockmayer.

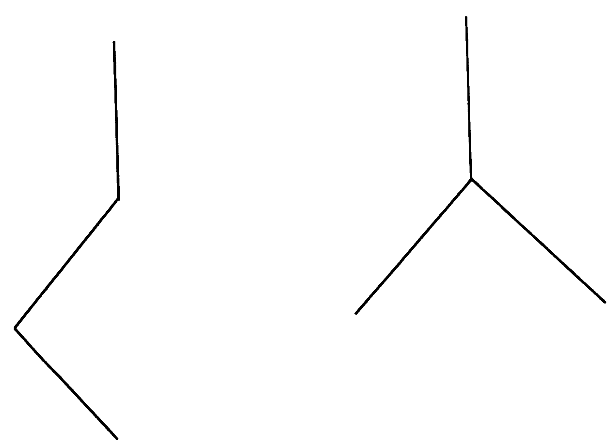

FIG. 3. The probabilities of these two figures are different if they are regarded as trees in a branching process, but are the same if they are regarded as polymers synthesized under the conditions assumed by Zimm and Stockmayer [17].

IV. OSCILLATIONS OF THE SANDPILE AND CRITICAL DIMENSION

If the process is subcritical or supercritical, the dimension of the corresponding trees is zero and infinity, respectively. For these models the infinitesimal probabilities of "death" and "reproduction" are μ and $\mu+\Delta$, respectively, where $\Delta<0$ for the subcritical case and $\Delta>0$ for the supercritical case. The corresponding Chapman-Kolmogorov equation was solved by Kendall in 1948 [19,13]. The limit corresponding to (6) is

$$\lim_{t\to\infty}\sum_{j=kt^{\alpha}}^{\infty}P'_{\Delta,1\to j}(t)=\left[1-\frac{1}{e^{\Delta t}\left[1+\frac{\mu}{\Delta}\right]-\frac{\mu}{\Delta}}\right]^{kt^{\alpha}-1}=\begin{cases}0, & \text{if } \Delta<0, \text{ for all } \alpha\\ 1, & \text{if } \Delta>0, \text{ for all } \alpha\ ,\end{cases} \tag{7}$$

which means that the dimension of the subcritical trees is zero and the dimension of the supercritical trees is infinite, and the same is true for the isotropic, homogeneous model.

What bearing do the present results have on the question of the critical dimension? The criticality in "self-organized criticality" is stable in the sense that accumulations or depletions of the quantity that is flowing through the system tend to be eliminated. Equivalently, the flux of that quantity tends to be kept constant. The precise mechanism by which this is achieved has been called "feedback" by Kadanoff (see [20]) and identified by Sornette [20] as an essential ingredient of self-organized criticality. For the branching process model, different feedbacks would correspond to different ways in which the addition or depletion of grains of sand would change the difference Δ. Some would be more natural than others, be it for their mathematical simplicity or for being derivable from simple geometrical considerations.

The picture that comes out of all this is that of a sandpile which oscillates around its critical slope. As the sandpile becomes bigger and bigger, the average dimension of the supercritical trees diverges. But, at the same time, the size of the fluctuations of the slope around its critical value goes (presumably, by virtue of some sort of central limit theorem) to zero. So the question of whether the average dimension of an avalanche that takes place in an infinitely large, oscillating sandpile is finite or not is answered by a $\infty.0$ limit. Different kinds of feedbacks might give different results. Of course, if no theorem similar to the central limit theorem applies to the average value of the slope of infinitely large avalanches, then the critical dimension is going to be infinite.

In any case, one should be aware that values for the critical dimension obtained from models of self-organized criticality that do not incorporate feedback could change, even by an infinite amount, when the oscillations around criticality are taken into consideration.

We have chosen particular branching process models to compute critical dimensions. However, other ways to implement subcriticality and supercriticality are also possible. We believe that our results depend only on whether the expected value, $m=\sum_{j=0}^{\infty}jP_j(t)$ is smaller than, greater than, or equal to 1.

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